Game Theory

Lecture 03:

Nash Equilibrium

Nash Equilibrium: When players' beliefs are correct

- Consider the two-player game in the figure:
  - All pure outcomes survive the iterated elimination of strictly dominated strategies.
  - Because there are only two players, this is also the set of rationalizable pure strategy profiles.

	l	*r*
U	**5,5**	**4,4**
D	**4,4**	**5,5**

- Let's see why the strategy profile (U,r) is rationalizable:

$\mathscr{R}\quad(U)$	$\mathscr{R}$ plays U,
$\mathscr{R}\quad\mathscr{C}\quad(l)$	$\mathscr{R}$ believes $\mathscr{C}$ will play l,
$\mathscr{R}\quad\mathscr{C}\quad\mathscr{R}\quad(U)$	$\mathscr{R}$ believes $\mathscr{C}$ believes $\mathscr{R}$ will play U.

$\mathscr{C}\quad(r)$	$\mathscr{C}$ plays r,
$\mathscr{C}\quad\mathscr{R}\quad(D)$	$\mathscr{C}$ believes $\mathscr{R}$ will play D,
$\mathscr{C}\quad\mathscr{R}\quad\mathscr{C}\quad(r)$	$\mathscr{C}$ believes $\mathscr{R}$ believes $\mathscr{C}$ will play r.

$\mathscr{R}$ (U)
$\mathscr{R}$ $\mathscr{C}$ (l)
$\mathscr{R}$ $\mathscr{C}$ $\mathscr{R}$ (U)

$\mathscr{C}$ (r)
$\mathscr{C}$ $\mathscr{R}$ (D)
$\mathscr{C}$ $\mathscr{R}$ $\mathscr{C}$ (r)

	l	*r*
U	**5,5**	**4,4**
D	**4,4**	**5,5**

- After this game is played,
  - each player will realize *ex post* that her beliefs about her opponent's play were incorrect and, further, each will regret her own choice in the light of what she learned about her opponent's strategy.
- $\mathcal{R}$ believed that $\mathcal{C}$ would play l, but $\mathcal{C}$ instead chose r.
  - Had $\mathcal{R}$ known that $\mathcal{C}$ would choose r, she would have chosen D instead.
- Similarly, $\mathcal{C}$ believed that $\mathcal{R}$ would play *D*, but $\mathcal{R}$ played U instead.
  - Had $\mathcal{C}$ known that $\mathcal{R}$ would play U, he would have preferred to have chosen l.

In this (U,r) outcome, then, each player was choosing a best response to her beliefs about the strategy of her opponent, **but each player's beliefs were wrong**!

Nash Equilibrium: When players' beliefs are correct

- Now consider the strategy profile (U,*l*):

$\mathcal{R}\ (U)$	$\mathcal{C}\ (l)$
$\mathcal{R}\ \mathcal{C}\ (l)$	$\mathcal{C}\ \mathcal{R}\ (U)$
$\mathcal{R}\ \mathcal{C}\ \mathcal{R}\ (U)$	$\mathcal{C}\ \mathcal{R}\ \mathcal{C}\ (l)$

	l	*r*
U	**5,5**	**4,4**
D	**4,4**	**5,5**

- When the game is played this way—viz. Row plays Up and Column plays left—**each player's prediction of her opponent's strategy was indeed correct**.
  - Since **each player was playing a best response to her correct beliefs**, neither player regrets her own choice of strategy.

When **rational players** correctly forecast the strategies of their opponents they are not merely playing best responses to their *beliefs* about their opponents' play; they are **playing best responses to the *actual* play of their opponents**.

When all players correctly forecast their opponents' strategies, and play best responses to these forecasts, the resulting strategy profile is a ***Nash equilibrium***.

Nash Equilibrium (NE)

A ***pure-strategy Nash equilibrium*** of a strategic-form game is a pure-strategy profile $\boldsymbol{s}^* \in \boldsymbol{S}$ such that "every player is playing a best response to the strategy choices of her opponents." More formally, we say that $\boldsymbol{s}^*$ is a Nash equilibrium if:

$$(\forall i \in I) \quad s_i^* \text{ is a best response to } \boldsymbol{s}_{-i}^*,$$

or, equivalently,

$$(\forall i \in I) \quad s_i^* \in \mathrm{BR}_i(\boldsymbol{s}_{-i}^*),$$

In NE, best-response correspondences intersect!

or, more notationally, $(\forall i \in I)\,(\forall s_i \in S_i) \quad u_i(s_i^*, \boldsymbol{s}_{-i}^*) \geq u_i(s_i, \boldsymbol{s}_{-i}^*).$

- Note that when a player *i* judges the optimality of her part of the equilibrium prescription, she does assume that her *opponents* will play their part $\boldsymbol{s}_{-i}^*$ of the prescription.

- Therefore, she is asking herself the question: Does there exist a ***unilateral* deviation** s_i for me such that I would ***strictly*** gain from such defection given that the opponents held truly to their prescriptions?

A game need not have a pure-strategy Nash equilibrium.

- Consider the matching pennies game:

Player 1 \ Player 2	heads	tails
heads	$(-1, 1)$	$(1, -1)$
tails	$(1, -1)$	$(-1, 1)$

- No matter how the players think the game will be played (i.e. what pure-strategy profile will be played), one player will always be distinctly unhappy with her choice and would prefer to change her strategy.

- This nonexistence problem when we restrict ourselves to pure strategies was historically a major motivation for the introduction of mixed strategies into game theory:

- We will see that the:

The existence of mixed-strategy Nash equilibrium *is* guaranteed.

Mixed-Strategy Nash Equilibrium.

Definition

A *Nash equilibrium* of a strategic-form game is a mixed-strategy profile $\boldsymbol{\sigma}^* \in \Sigma$ such that "every player is playing a best response to the strategy choices of her opponents." More formally, we say that $\boldsymbol{\sigma}^*$ is a Nash equilibrium if

$$(\forall i \in I) \quad \sigma_i{}^* \text{ is a best response to } \boldsymbol{\sigma}_{-i}{}^*,$$

or, equivalently,

$$(\forall i \in I) \quad \operatorname{supp} \sigma_i{}^* \subset \mathrm{BR}_i(\boldsymbol{\sigma}_{-i}{}^*),$$

or, more notationally,

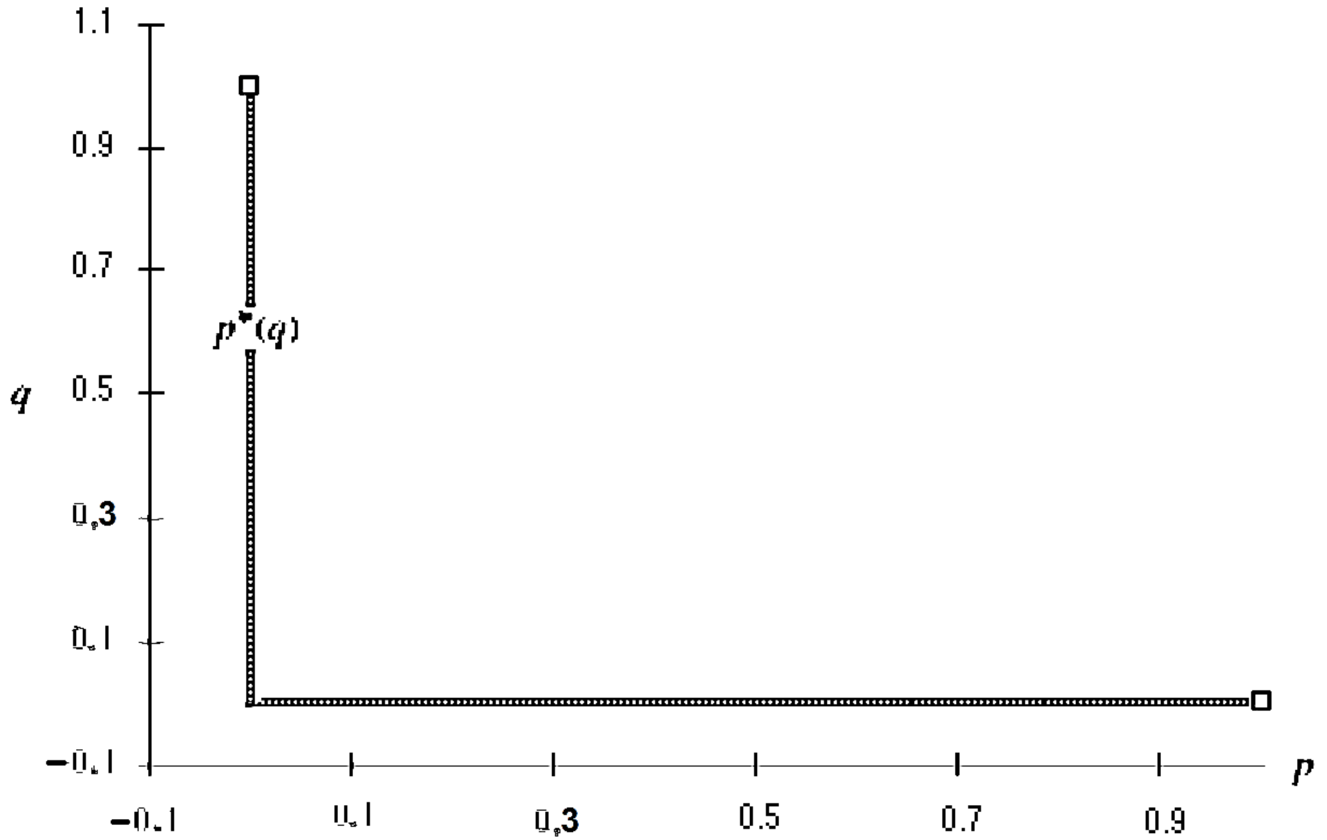
$$(\forall i \in I)\ (\forall s_i \in S_i) \quad u_i(\sigma_i{}^*, \boldsymbol{\sigma}_{-i}{}^*) \geq u_i(s_i, \boldsymbol{\sigma}_{-i}{}^*).$$

Example: Mixed NE

	l:[q]	r:[1-q]
U:[p]	1,1.5	3,1
D:[1-p]	4,2	3,3

$$U_R(U;q) = 1 \times q + 3 \times (1-q) = 3 - 2q$$
$$U_R(D;q) = 4 \times q + 3 \times (1-q) = q + 3$$

$$D \succcurlyeq_{\mathcal{R}} U \Leftrightarrow U_R(D;q) \geq U_R(U;q)$$
$$q + 3 \geq 3 - 2q \Rightarrow q \geq 0$$



Example (cont'd)

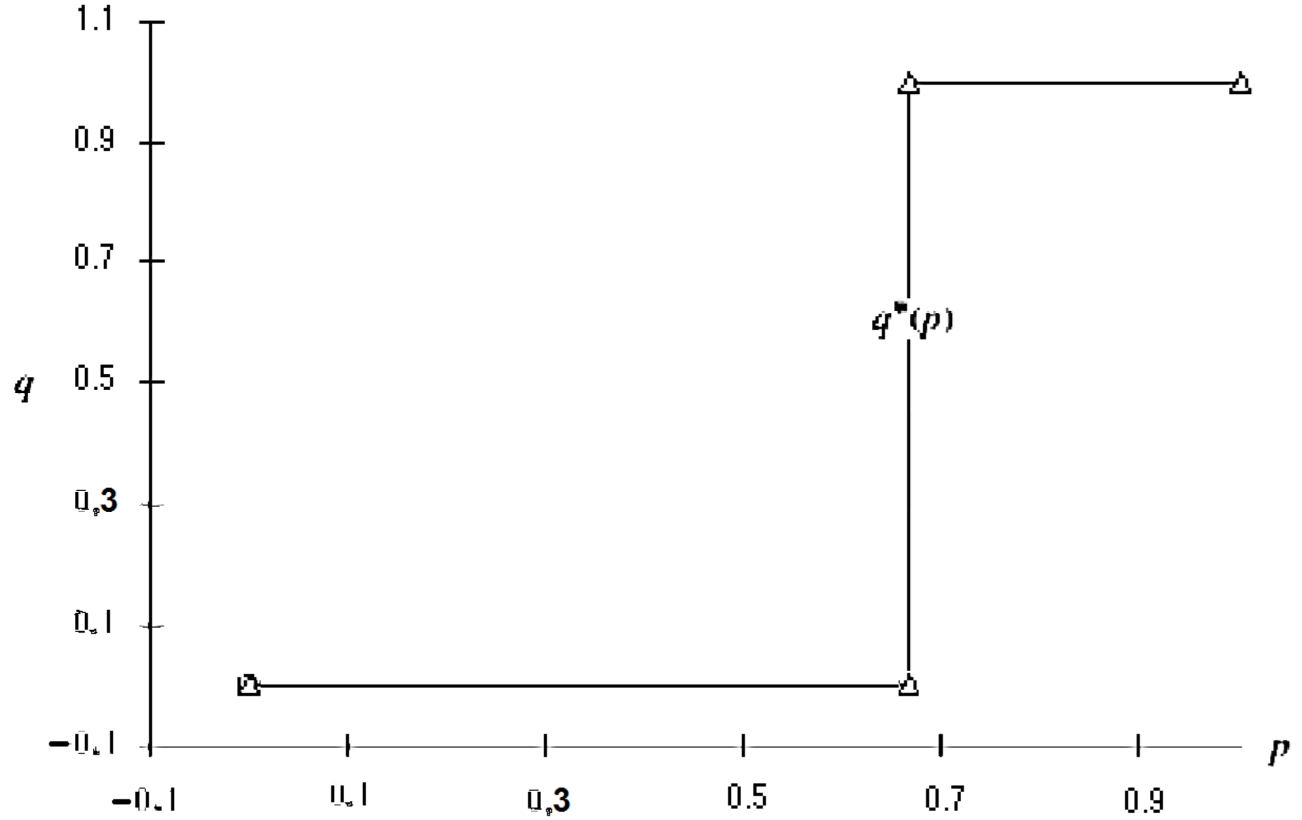
	l:[q]	*r*:[1-q]
U:[p]	1,1.5	3,1
D:[1-p]	4,2	3,3

$$U_C(p;l) = 1.5 \times p + 2 \times (1-p) = 2 - 0.5\text{p}$$

$$U_C(p;r) = 1 \times p + 3 \times (1-p) = 3 - 2p$$

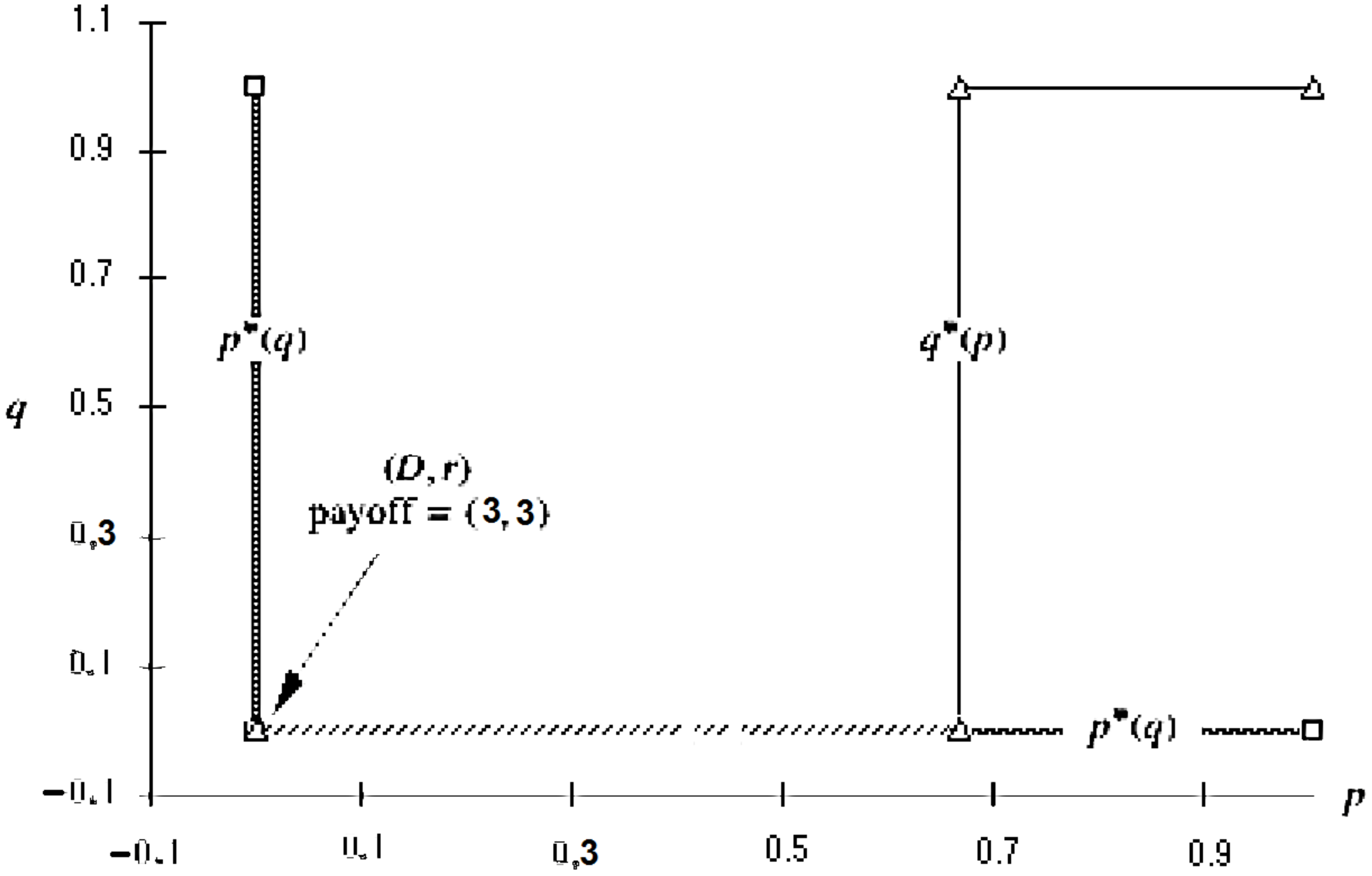
$$l \succcurlyeq_C r \Leftrightarrow U_C(l;p) \geq U_C(r;p)$$

$$2 - 0.5p \geq 3 - 2p \Rightarrow p \geq \frac{2}{3}$$



- The intersection of the graphs of A's and B's best-response correspondences is a line segment along which **B plays q=0 and A mixes with any probability p on [0,2/3]**.
- We note that the unique pure-strategy Nash equilibrium we identified earlier is the left endpoint of this set.

	l:[q]	*r*:[1-q]
U:[p]	1,1.5	3,1
D:[1-p]	4,2	3,3



This is a game with a continuum of equilibria!

Pareto Optimality

- We've defined some canonical games, and thought about how to play them. Now let's examine the games from the outside
- From the point of view of an outside observer, can some outcomes of a game be said to be better than others?
  - can't say one agent's interests are more important than another's
  - imagine trying to find the revenue-maximizing outcome when you don't know what currency is used to express each agent's payoff
- Are there ways to still prefer one outcome to another?

Pareto Optimality

- Idea: sometimes, one outcome o is at least as good for every agent as another outcome o', and there is some agent who strictly prefers o to o'
  - in this case, it seems reasonable to say that o is better than o'
  - we say that o Pareto-dominates o'.

$o: (7, 8)$
$o': (7, 2)$

Definition (Pareto Optimality)

An outcome o^* is Pareto-optimal if there is no other outcome that Pareto-dominates it.

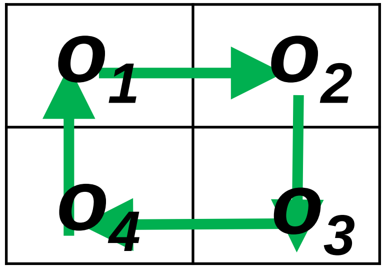
- can a game have more than one Pareto-optimal outcome?

Yes:

1,1	1,1
1,1	1,1

- does every game have at least one Pareto-optimal outcome?

Yes:



Pareto Optimal Outcomes in Example Games

NE

PO

	Left	Right
Left	1, 1	0, 0
Right	0, 0	1, 1

	B	F
B	2, 1	0, 0
F	0, 0	1, 2

	Heads	Tails
Heads	1, −1	−1, 1
Tails	−1, 1	1, −1

	C	*D*
C	−1, −1	−4, 0
D	0, −4	−3, −3

The paradox of Prisoner's dilemma:

the Nash equilibrium is the only non-Pareto-optimal outcome!

Nash equilibria can be vulnerable to multiplayer deviations

- The definition of NE only requires the absence of any profitable unilateral deviations by any player.
- A Nash equilibrium is not guaranteed to be invulnerable to deviations by coalitions of players however.

There are two pure-strategy NE: (U,l,A) and (D,r,B) where: (U,l,A) Pareto dominates (D,r,B).

A:

	l	*r*
U	**0,0,10**	**-5,-5,10**
D	**-5,-5,0**	**1,1,-5**

B:

	l	*r*
U	**-2,-2,0**	**-5,-5,0**
D	**-5,-5,0**	**-1,-1,5**

- Consider the (U, *l*,A) equilibrium. No player wants to deviate unilaterally.
- Now, fix Matrix's choice at A and consider the joint deviation by Row and Column from (U, *l*) to (D,r). Both would profit from such a shift in their strategies, yet (U,*l*,A) is still a Nash equilibrium.

A strategy profile is a ***strong equilibrium*** if no coalition (including the grand coalition, i.e. all the players collectively) can profitably deviate from the prescribed profile.

Strong Equilibria: Coalition-Proof Equilibria

- By definition, **any strong equilibrium is both Pareto optimal and a Nash equilibrium**.
- A strong equilibrium need not exist!
- Also note that (D,r,A) to which the coalition of Row and Column might defect is itself not even a Nash equilibrium.
  - ➢ Therefore one could question whether it should be used as the basis for rejecting (U,*l*,A).

	l	*r*
U	0,0,10	-5,-5,10
D	-5,-5,0	1,1,-5

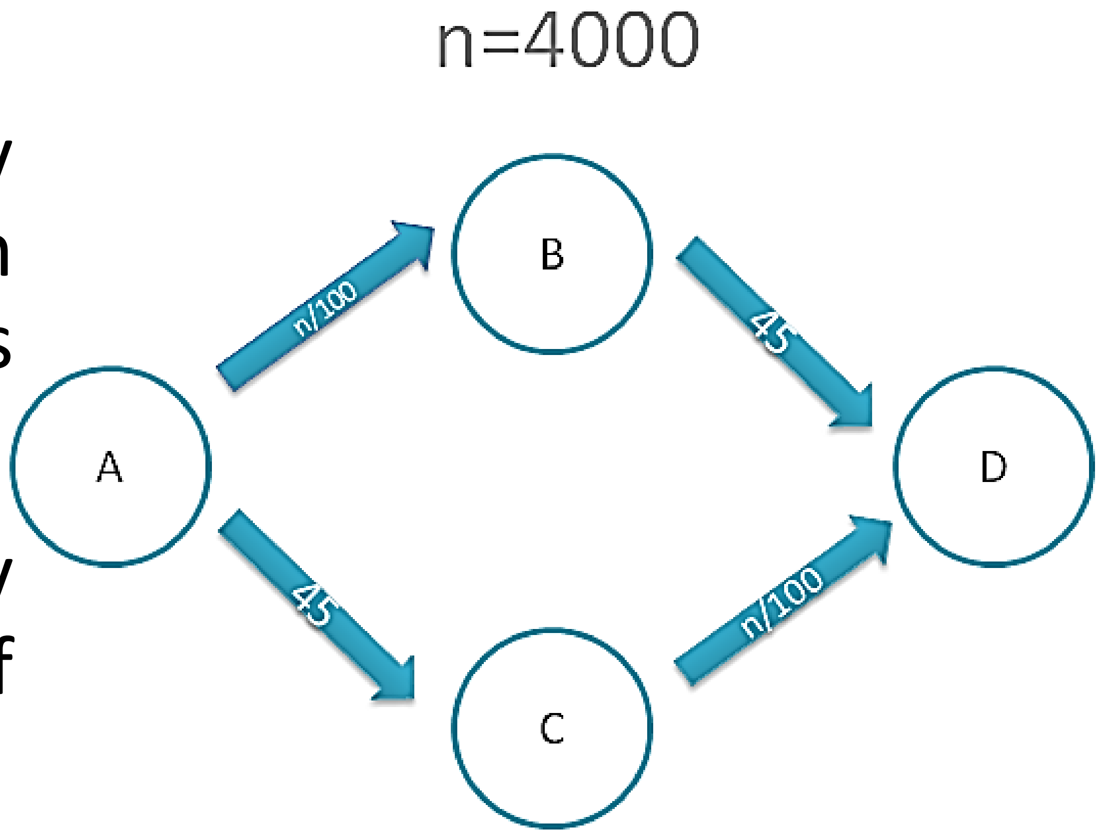
A

	l	*r*
U	-2,-2,0	-5,-5,0
D	-5,-5,0	-1,-1,5

B

Selfish Routing & Braess' Paradox

- Suppose there are 4000 commuters who have to travel from **A -> D**.
- The paths **A->B** and **C->D** are very narrow and the time it takes for n persons to travel through them is given by: **n/100 minutes**.
- The paths **B->D** and **A->C** are very broad and it takes a constant time of **45 minutes** to travel through them.



- **Which route should they take?**

Selfish Routing & Braess' Paradox

- A immediate solution is that **2000 take the route: A->B->D** and the other **2000 take the route A -> C -> D**. Why?
- If 1 of them decides, say the one who was earlier taking the route

  A -> B -> D, to shift to **A->C->D** then:

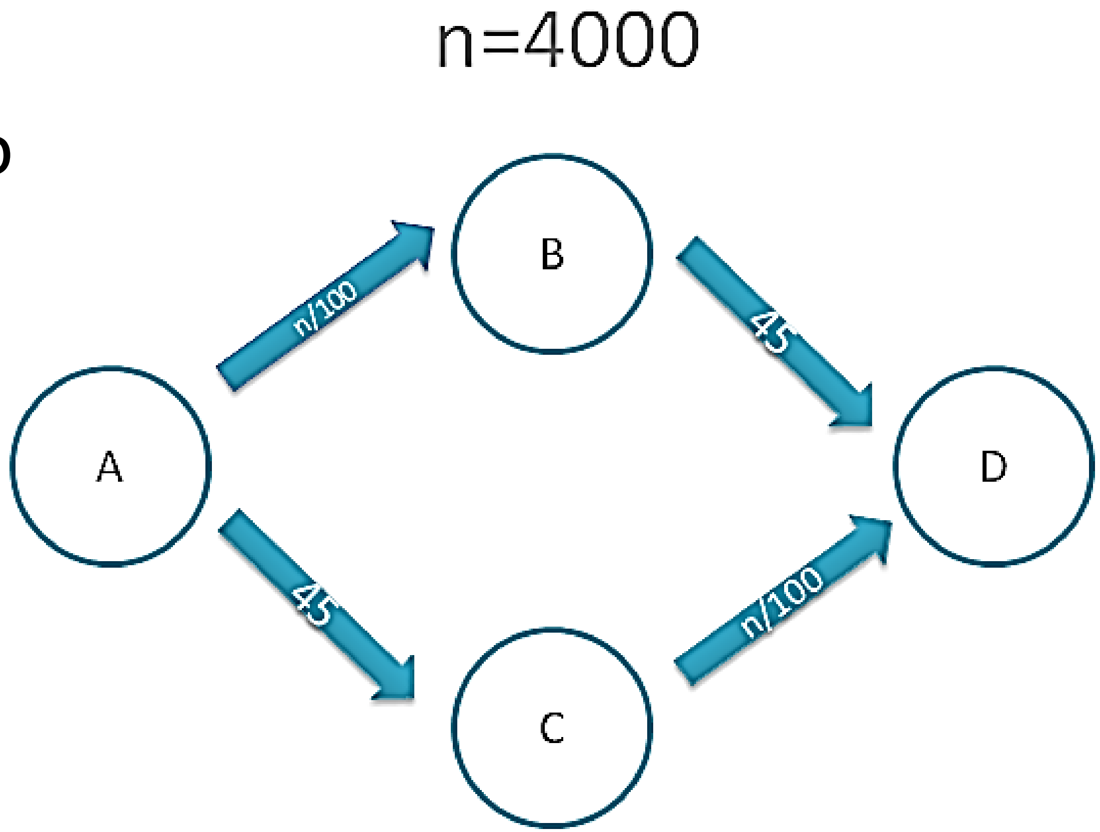
  the time now takes him to travel:

  45 + 2001/100 = 65.01 mins,

  while earlier it took him only

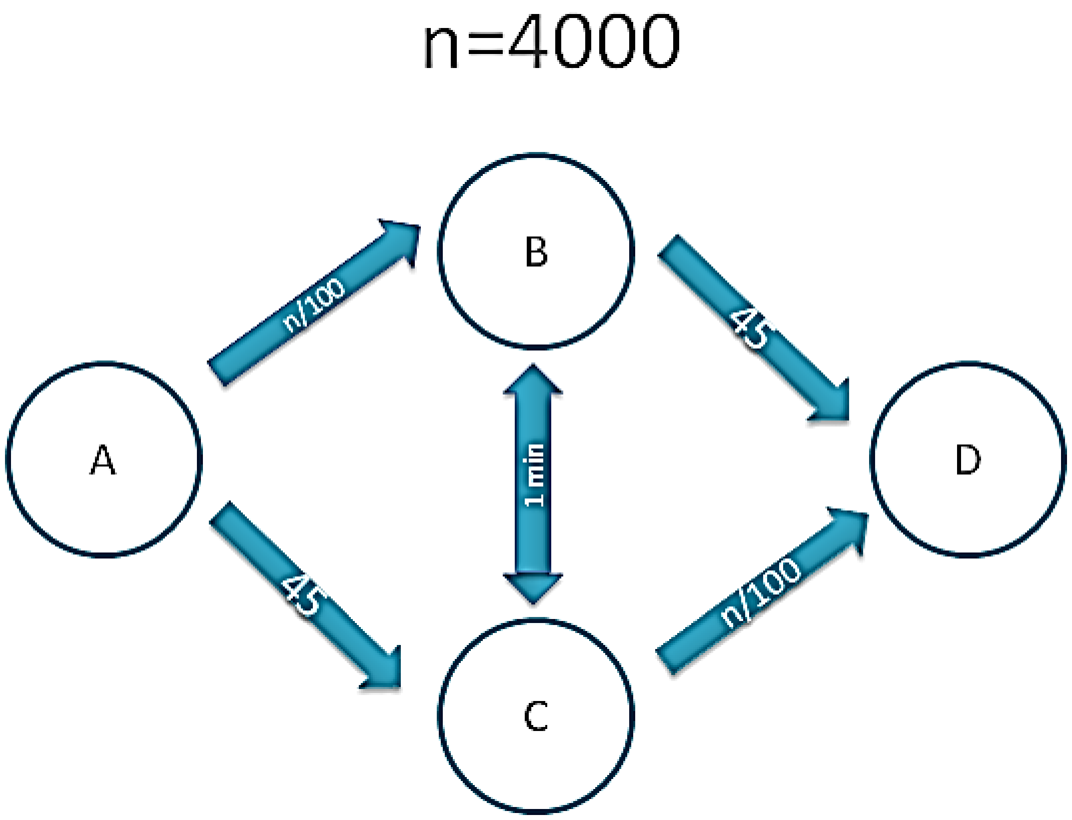
  45 + 2000/100 = 65 mins.
- So why should he deviate ? No reason! He will not and this is a:

Nash equilibrium (NE)



Selfish Routing & Braess' Paradox

- Now consider that being a good transport minister you build one two -way road BC, which is so wide that it takes only a constant time of **1 min** to go from **B->C** or **C->B**.
- Now, what path will the commuters take?
- This time all 4000 of them will take the path: **A->B->C->D**.
- The Time taken by each person: **4000/100 + 1 + 4000/100 = 81 mins**. **Why?!**



Selfish Routing & Braess' Paradox

- If one of them decides to deviate and takes the path (w.l.g.):

A->B->D,

- Then, the total time it takes for him:

4000/100 + 45 = 85 mins.

- If he takes the path:

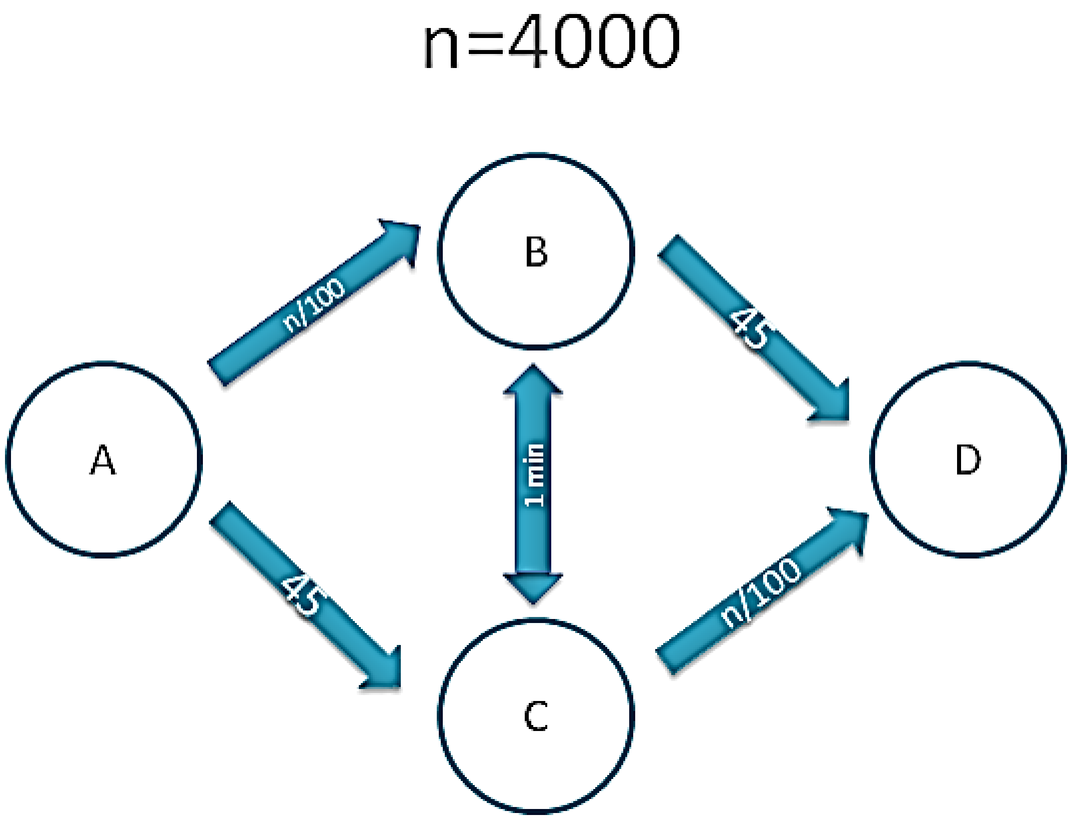
A->C->B->D,

- Then the total time it takes for him:

45+ 1 + 45 = 91 mins.

- So no one will deviate and

A->B->C->D is a Nash Equilibrium.



As you can see building an extra path B <->C increased the overall commuting time. Though this is counter-intuitive, try to think it through!

Selfish Routing & Braess' Paradox

- **Now, the interesting fact**:
- Many bridges/roads have been closed/broken because of this phenomenon known as **Braess Paradox**.
  - In **Seoul, South Korea**, a speeding-up in traffic around the city was seen when a motorway was removed as part of the Cheonggyecheon restoration project.
  - In **Stuttgart, Germany** after investments into the road network in 1969, the traffic situation did not improve until a section of newly built road was closed for traffic again.
  - In 1990 the closing of **42nd street in New York City** reduced the amount of congestion in the area.
  - In 2008 Youn, Gastner and Jeong demonstrated specific routes in **Boston, New York City and London** where this might actually occur and pointed out roads that could be closed to reduce predicted travel times.

Selfish Routing & Braess' Paradox

- **Braess's paradox** was discovered in 1968 by mathematician **Dietrich Braess**.
- He noticed that adding a road to a congested road traffic network could increase overall journey time, and it has been used to explain instances of improved traffic flow when existing major roads are closed.
- His idea was that if each driver is making the **optimal self-interested decision** as to which route is quickest, a shortcut could be chosen too often for drivers to have the shortest travel times possible.
- More formally, the idea behind Braess' discovery is that the **Nash equilibrium may not equate with the best overall flow through a network**!

Selfish Routing & Braess' Paradox

- Adding extra capacity to a network when the moving entities selfishly choose their route can in some cases reduce overall performance.

- That is because the Nash equilibrium of such a system is not necessarily optimal.

- The paradox may have analogues in **electrical power grids** and **biological systems**.

Tragedy of the commons

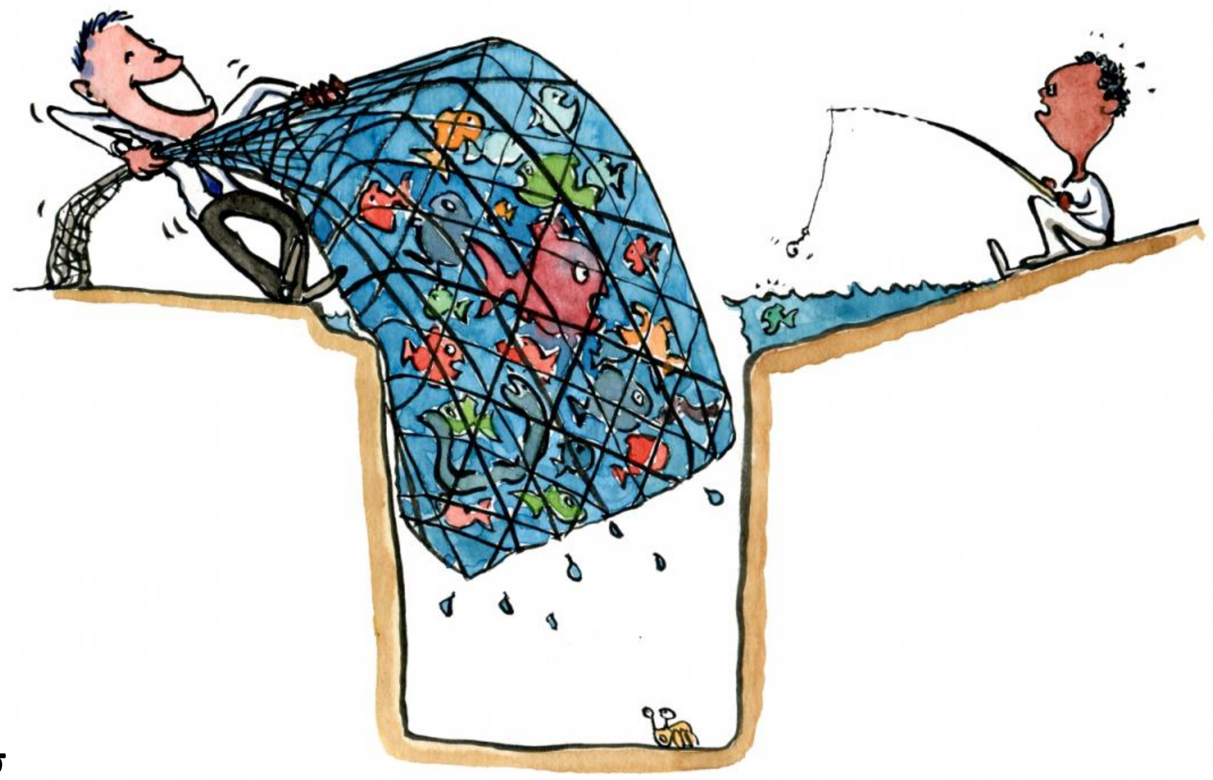
first described by Garrett Hardin in a 1968 article in Science. Briefly, it says that **a shared resource is inevitably ruined by uncontrolled use.**

- Assume there are n players (fishers, fishing firms, countries, groups of countries) harvesting a common fish resource x

- Each player maximises her own economic gains from the resource by choosing a fishing effort $s_i \in [0,1]$.

  - This means that each player chooses her optimal startegy taking into account other players' strategy

Non-cooperative fisheries game



Fisheries Game: Building Objective Functions of the Players

- The size of the fish stock at time t is denoted by $x(t)$, which evolves over time according to:

$$\frac{dx}{dt} = F(x) - \sum_{i=1}^{n} h_i$$

Agent i's catch at time t

$$h_i = s_i x$$

Logistic growth function
(See study.com)

$$F(x) = Rx\left(1 - \frac{x}{K}\right)$$

- For simplicity, assume a steady state ($\frac{dx}{dt}=0$, R=K=1 ⬄ standard logistic):

Stock biomass depends on all fishing efforts

$$x = \left(1 - \sum_{i=1}^{n} s_i\right)$$

- Players maximize their catch h_i from the fishery:

Objective function Of player i

$$p_i(s) := \begin{cases} s_i\left(1 - \sum_{j=1}^{n} s_j\right) & \text{if } \sum_{j=1}^{n} s_j \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Tragedy of the commons (Cont'd)

- We can write the payoff function in a more compact way as:

$$p_i(s) := \max(0, s_i(1 - \sum_{j=1}^{n} s_j)).$$

To find a Nash equilibrium of this game, fix $i \in \{1, \ldots, n\}$ and s_{-i} and denote $\sum_{j \neq i} s_j$ by f. Then $p_i(s_i, s_{-i}) = \max(0, s_i(1 - f - s_i))$.

By elementary calculus player i's payoff becomes maximal when $\frac{dp_i}{ds_i} = 0$ *and* $\frac{d^2 p_i}{ds_i^2} < 0$

$$\frac{dp_i}{ds_i} = 1 - f - 2s_i = 0 \;\Rightarrow\; s_i^* = \frac{1-f}{2} = \frac{1 - \sum_{j \neq i} s_j^*}{2} \;\Rightarrow\; s_i^* = 1 - \sum_{j=1}^{n} s_j^*$$

This action is the best response for player *i* given $\boldsymbol{s}_{-i}^*$ for i=1,2,…, n then s^* is a Nash equilibrium.

This system of n linear equations has a unique solution

Summing over all players: $\sum_{i=1}^{n} s_i^* = n - n\sum_{j=1}^{n} s_j^* \Rightarrow \sum_{i=1}^{n} s_i^* = \frac{n}{n+1} \Rightarrow s_i^* = 1 - \frac{n}{n+1} = \frac{1}{n+1}$ for $i \in \{1, \ldots, n\}$.

Tragedy of the commons (Cont'd)

- Fishery game's NE: $s_i = \frac{1}{n+1}$ for $i \in \{1, \ldots, n\}$.

In this strategy profile each player's payoff is $\frac{1-n/(n+1)}{n+1} = \frac{1}{(n+1)^2}$,

so its social welfare is $\frac{n}{(n+1)^2}$.

There are other Nash equilibria. Indeed, suppose that for all $i \in \{1, \ldots, n\}$ we have $\sum_{j \neq i} s_j \geq 1$, which is the case for instance when $s_i = \frac{1}{n-1}$ for $i \in \{1, \ldots, n\}$.

- It is straightforward to check that each such strategy profile is a Nash equilibrium in which each player's payoff is 0 and hence the social welfare is also 0.
- It is easy to check that no other Nash equilibria exist.

Tragedy of the commons (Cont'd)

To find a strategy profile in which social optimum is reached fix a strategy profile s and let $f := \sum_{j=1}^{n} s_j$.

First note that if $f > 1$, then the social welfare is 0.

So assume that $f \leq 1$. Then $\sum_{j=1}^{n} p_j(s_j) = f(1-f)$.

By elementary calculus this expression becomes maximal precisely when $f = \frac{1}{2}$ and then it equals $\frac{1}{4}$.

Comparing with the social welfare of the NE

for all $n > 1$ we have $\frac{n}{(n+1)^2} < \frac{1}{4}$.

So the social welfare of each solution for which $\sum_{j=1}^{n} s_j = \frac{1}{2}$ is superior to the social welfare of the Nash equilibria.

In particular, no such strategy profile is a Nash equilibrium.

Tragedy of the commons (Conclusion)

In conclusion, the social welfare is maximal, and equals $\frac{1}{4}$, when precisely half of the common resource is used.

In contrast, in the 'best' Nash

the social welfare is $\frac{n}{(n+1)^2}$ and the fraction $\frac{n}{n+1}$ of the common resource is used.

So when the number of players increases, the social welfare of the best Nash equilibrium becomes arbitrarily small, while the fraction of the common resource being used becomes arbitrarily large.

The analysis carried out reveals that for the adopted payoff functions **the common resource will be overused, to the detriment of the players (society)**.

The same conclusion can be drawn for a much larger of class payoff functions that properly reflect the characteristics of using a common resource.

Price of Anarchy (PoA)

- The **Price of Anarchy (PoA)** is a concept in economics and game theory that measures how the efficiency of a system degrades due to selfish behavior of its agents.

Consider a game $G = (N, S, u)$, defined by a set of players N, strategy sets S_i for each player and utilities $u_i : S \to \mathbb{R}$ (where $S = S_1 \times \ldots \times S_n$ also called set of outcomes). We can define a measure of efficiency of each outcome which we call welfare function $W : S \to \mathbb{R}$. Natural candidates include the sum of players utilities (utilitarian objective) $W(s) = \sum_{i \in N} u_i(s)$, minimum utility (fairness or egalitarian objective) $W(s) = \min_{i \in N} u_i(s)$, ..., or any function that is meaningful for the particular game being analyzed and is desirable to be maximized.

PoA (cont'd)

We can define a subset $E \subseteq S$ to be the set of strategies in equilibrium (for example, the set of Nash equilibria). The Price of Anarchy is then defined as the ratio between the optimal 'centralized' solution and the 'worst equilibrium':

$$PoA = \frac{\max_{s \in S} W(s)}{\min_{s \in E} W(s)}$$

If, instead of a 'welfare' which we want to 'maximize', the function measure efficiency is a 'cost function' $C : S \to \mathbb{R}$ which we want to 'minimize' (e.g. delay in a network) we use (following the convention in approximation algorithms):

$$PoA = \frac{\max_{s \in E} C(s)}{\min_{s \in S} C(s)}$$

Price of Stability (PoS)

A related notion is that of the **Price of Stability** (**PoS**) which measures the ratio between the 'best equilibrium' and the optimal 'centralized' solution:

$$PoS = \frac{\max_{s\in S} W(s)}{\max_{s\in E} W(s)}$$

or in the case of cost functions:

$$PoS = \frac{\min_{s\in E} C(s)}{\min_{s\in S} C(s)}$$

We know that $1 \leq PoS \leq PoA$ by the definition. It is expected that the loss in efficiency due to game-theoretical constraints is somewhere between 'PoS' and 'PoA'.

Example

Prisoner's dilemma

Consider the 2x2 game called prisoner's dilemma, given by the following cost matrix:

	Cooperate	Defect
Cooperate	1, 1	7, 0
Defect	0, 7	5, 5

and let the cost function be $C(s_1, s_2) = u_1(s_1, s_2) + u_2(s_1, s_2)$. Now, the minimum cost would be when both players cooperate and the resulting cost is $1 + 1 = 2$. However, the only Nash equilibrium occurs when both defect, in which case the cost is $5 + 5 = 10$. Thus the Price of Anarchy of this game will be $10/2 = 5$.

Example: Job Scheduling

There are N players and each of them has a job to run.

They can choose one of M machines to run the job.

Each machine has a speed $s_1, \ldots, s_M > 0.$

Each job has a weight $w_1, \ldots, w_N > 0.$

A player picks a machine to run his or her job on.

So, the strategies of each player are $A_i = \{1, 2, \ldots, M\}$.

Define the *load* on machine j to be: $$L_j(a) = \frac{\sum_{i:a_i=j} w_i}{s_j}.$$

The cost for player i is $c_i(a) = L_{a_i}(a)$, i.e., the load of the machine they chose.

We consider the egalitarian cost function

$$\mathrm{MS}(a) = \max_j L_j(a),$$ here called the *makespan.*

Definition *Lexicographic Sort*

If L_i *is the load of machine i, the vector:* $(L_1, \ldots, L_M) < (L'_1, \ldots, L'_M)$

if until a certain index i the loads are equal, and at index i $L_i < L'_i$

A configuration $\boldsymbol{a}$ *is said to be less than* $\boldsymbol{a}'$ *if:*

the load vector associated with $\boldsymbol{a}$ *is less than that of* $\boldsymbol{a}'$.

We would like to take a socially optimal action profile $\boldsymbol{a}^*$.

This would mean simply an action profile whose makespan is minimum.

There may be several such action profiles leading to a variety of different loads distributions (all having the same maximum load).

Among these, we further restrict ourselves to one that has a minimum second-largest load.

Again, this results in a set of possible load distributions, and we repeat until the Mth-largest load, where there can only be one distribution of loads (unique up to permutation).

Example: Job Scheduling (Cont'd)

Claim. For each job scheduling game, there exists at least one pure-strategy Nash equilibrium.

Proof. the *lexicographic* smallest sorted load vector is a pure-strategy NE.

Reasoning by contradiction, suppose that

some player i could strictly improve by moving from machine j to machine k.

This means that the increased load of machine k after the move is still smaller than the load of machine j before the move.

As the load of machine j must decrease as a result of the move and no other machine is affected, this means that the new configuration is guaranteed to have reduced the jth-largest (or higher ranked) load in the distribution.

This, however, violates the assumed lexicographic minimality of a.

Q.E.D.

Claim. For each job scheduling game, the pure PoA is at most M.

Proof: Let $s^* = \max_j s_j$. In the worst case any *Nash equilibrium* is bounded by:

$$MS(a) \leq \frac{\sum_{i=1}^{n} w_i}{s^*} = W$$

(Otherwise, a player that observes a higher load than W can move to a machine with speed s^* for which its load after the migration is always less than *W*).

We also have that $$MS(a) \geq \frac{\sum_{j=1}^{n} w_j}{\sum_{i=1}^{M} s_i}.$$

(Which is the case if we can distribute each player's weight in equally over all machines).

Using the above bounds, we get:

$$PoA \leq \frac{\sum_{i=1}^{n} w_i / s^*}{\sum_{i=1}^{n} w_i / \sum_{j=1}^{M} s_j} = \frac{\sum_{j=1}^{M} s_j}{s^*} \leq M$$

Since $S_j \leq S^*$, for every machine j.